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This investigation is concerned with the dynamic simulation of complex structures consisting of an assemblage of substructures. Each of the substructures possesses a large number of degrees of freedom. In fact, in theory a continuous substructure has an infinite number of degrees of freedom, but for practical reasons each of the substructures must be simulated by a limited number of degrees of freedom. The component mode synthesis method advocates representing the substructure motion by a given number of substructure modes. In an earlier paper, the authors of this report have argued that component modes are actually not necessary and

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DYNAMIC SIMULATION OF COMPLEX STRUCTURES

FIRST INTERIM TECHNICAL REPORT

L. MEIROVITCH AND A. L. HALE

NOVEMBER 1978

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Contents

Abstract

This investigation is concerned with the dynamic simulation of complex structures consisting of an assemblage of substructures. Each of the substructures possesses a large number of degrees of freedom. In fact, in theory a continuous substructure has an infinite number of degrees of freedom, but for practical reasons each of the substructures must be simulated by a limited number of degrees of freedom. The component mode synthesis method advocates representing the substructure motion by a given number of substructure modes. In an earlier paper, the authors of this report have argued that component modes are actually not necessary and admissible functions suffice. Equally important is the treatment of discrete substructures with a large number of degrees of freedom. To simulate the motion of such substructures, these authors are advancing the concept of "admissible vectors". The first phase of the research has concentrated on the development of this concept and placing it on a sound mathematical foundation as well as developing methods for producing such vectors. To introduce the above ideas, a discrete model consisting of a rotating lumped-parameter cantilever beam has been used. Such a model is of special interest, as it can be regarded as simulating a helicopter blade.

1. Introduction

This investigation is concerned with the dynamic simulation of complex structures consisting of an assemblage of substructures. Each of the substructures possesses a large number of degrees of freedom. In fact, in theory a continuous substructure has an infinite number of degrees of freedom, but for practical reasons each of the substructures must be simulated by a limited number of degrees of freedom. The component mode synthesis method (Ref. 1) advocates representing the substructure motion by a given number of substructure modes. This seems to point to the necessity of producing a suitable set of modes, perhaps by solving the eigenvalue problem associated with the substructure. Quite often, however, this is no easy matter and in some cases the eigenvalue problem cannot even be defined properly.

In an earlier paper, the authors of this report have argued that component modes are actually not necessary to represent a substructure motion. Indeed, if the substructure is represented by a distributed-parameter mathematical model, then admissible functions suffice (Refs. 2-4). Equally important is the treatment of discrete substructures with a large number of degrees of freedom. To simulate the motion of such substructures, these authors are advancing the concept of "admissible vectors".

The first phase of the research work has concentrated on the development of the concept of admissible vectors for discrete systems, placing the concept on a sound mathematical foundation as well as developing methods for producing such vectors. To this end, the analogy with continuous systems has been invoked extensively. In discussing continuous systems, special emphasis has been placed on developing criteria for the selection of admissible functions capable of an accurate representation of

the motion of continuous members, as well as on developing methods for producing such admissible functions. In regard to the latter, the integral formulation of the eigenvalue problem (Ref. 5), which contains the Green's function in the kernel, proves to be a useful device for building in the smoothness and boundary conditions requirements.

Whereas the analogy between discrete and continuous systems makes physical sense, there are many mathematical aspects requiring close scrutiny. For the most part, they are related to differences between infinite and finite vector spaces. In some ways, the formulation for discrete systems is simpler, but the two mathematical models, discrete and distributed, cannot be divorced entirely. Indeed, one must consider the origin of the discrete system before being able to impose proper requirements on the smoothness and end conditions of the admissible vectors.

To introduce the above ideas, a discrete model consisting of a rotating lumped-parameter cantilever beam has been used. Such a model is of special interest, as it can be regarded as simulating a helicopter blade. Because the continuous counterpart of the model considered is a rotating distributed-parameter cantilever beam in bending, the smoothness requirements and geometric end conditions are more stringent than those for a set of lumped masses simulating a bar in axial vibration. Once again drawing on the analogy with continuous systems, ways for producing admissible vectors have been explored. One way is simply to discretize admissible functions. Based on the recognition that the discrete counterpart of the Green's function in the integral formulation is the flexibility influence coefficient, another way is to use the system flexibility matrix to generate vectors with the desired smoothness

and end conditions.

Preliminary results fully justify the concept of admissible vectors as a valid tool in dynamic synthesis. A comparison of eigenvalues and eigenvectors obtained by using different sets of admissible vectors proves quite encouraging. It also enhances the understanding of the problem and points the way to future directions of the research.

2. Mathematical Preliminaries for the Eigenvalue Problem for Continuous Systems

We shall be concerned with the eigenvalue problem described by the differential equation

$$Lu = \lambda Mu \tag{1}$$

where λ is a parameter, L is a linear homogeneous differential operator of order 2p, and M is merely a function of the spatial variables. Equation (1) must be satisfied by the function u at every point of an open domain D. Associated with the differential equation (1) there are p boundary conditions that u must satisfy at every point of the boundary S of the domain D. We assume that the boundary conditions are of the type

$$B_i u = 0 \quad , \quad i = 1,2,\ldots,p \tag{2}$$

where B_i are linear homogeneous differential operators involving derivatives normal to the boundary and along the boundary through order 2p-1. The eigenvalue problem consists of determining the values of the parameter λ for which there are nontrivial functions u satisfying the differential equation and the boundary conditions. The parameters λ are called eigenvalues and the corresponding functions u are called eigenfunctions (Ref. 5).

Quite often continuous systems lead to eigenvalue problems of the type defined by Eqs. (1) and (2) that do not lend themselves to

closed-form solution. In such cases we must be content with approximate solutions of the eigenvalue problem. We shall consider the Rayleigh-Ritz method for obtaining such approximate solutions. Before discussing the Rayleigh-Ritz method, we shall present certain mathematical preliminaries.

Let us consider a function u that is continuous in the closed domain $\bar{D}=D+S$. The function is assumed to be quadratically integrable in the Lebesgue sense, which implies that the integral $\|u\|^2 = (u,u) = \int_D u^2 dD$ exists, where $\|u\|$ is known as the norm of u. Our interest lies in approximating the function u by some other function. To this end, we consider the sequence of functions $u_1, u_2, \ldots, u_N, \ldots$ defined over D. Then, if $\lim_{N\to\infty} \|u_N-u\| = 0$, it is said that the sequence u_N converges in the mean to u.

Next, let us assume that the function u can be represented in the domain D by the series $u = \sum_{j=1}^{\infty} a_j\phi_j$, which is convergent in the mean, and consider the partial sum

$$u_N = \sum_{j=1}^{N} a_j\phi_j \tag{3}$$

Then, the set of functions ϕ_j is said to be complete if it is possible to find an integer N and a set of coefficients $a_1, a_2, \ldots, a_N$ such that for any $\varepsilon>0$, $\|u-u_N\| < \varepsilon$. The question remains as to the nature of the functions ϕ_j $(j=1,2,\ldots)$. To answer this question, we wish to delve a little deeper into the properties of the differential operator L. The operator L is said to be self-adjoint if for any two functions u and v in the field of definition of L

$$(Lu,v) = \int_D vLu dD = \int_D uLv dD = (u, Lv) \tag{4}$$

Integrating by parts, and considering the boundary conditions of the problem, we can write

$$\int_D vLudD = \int_D vA_D udD + \int_S vA_S udS \tag{5}$$

where A_D and A_S are "two-sided" symbolic operators symmetric in u and v. We should point out that, whereas L is an actual differential operator, A is an operator only in a symbolic sense. For convenience, we shall use the notation

$$[u, v]_A = \int_D vA_D udD + \int_S vA_S udS \tag{6}$$

The operators L and A are said to be positive definite if the inequality $[u,u]_A > 0$ is satisfied.

In our particular case, $[u,u]_A$ defines twice the potential energy associated with a given closed domain $\bar{D}$. Then, the quantity $\|u\|_A$ defined by $\|u\|_A^2 = [u,u]_A$ is known as the <u>energy norm</u> of u. When it is clear that the norm involves the operator A, the subscript A can be ignored. The sequence of functions $u_1, u_2, \ldots, u_N, \ldots$ <u>converges in energy</u> to u if $\lim_{N\to\infty} \|u_N - u\| = 0$. If u_N represents the partial sum (3), then if for any $\varepsilon > 0$ it is possible to find an integer N and a set of coefficients $a_1, a_2, \ldots, a_N$ such that $\|u - u_N\| < \varepsilon$, the set of functions $\phi_1, \phi_2, \ldots, \phi_N, \ldots$ is said to be <u>complete in energy</u>.

To generalize these ideas, let L be a positive definite operator defined on some Hilbert space H. But the scalar product (6) also defines a Hilbert space. The Hilbert space defined by the scalar product (6) is called complete if any sequence ϕ_j of its elements satisfying the condition $\lim_{k,n\to\infty} \|\phi_k - \phi_n\| = 0$ has a limit which is in the space; otherwise it is called incomplete. In general, it is incomplete and must be made complete by adding certain new elements to it. The new Hilbert space thus constructed is called the <u>energy space</u> and is denoted by H_A. Hence, the functions ϕ_j must belong to the space H_A. If they belong also to the domain of existence of L, then $[\phi_i, \phi_j] = (L\phi_i, \phi_j)$. The question is as to what distinguishes the functions belonging to H from

those belonging to H_A. It is clear that the integration by parts leading to Eq. (5) lowers the order of the operator involved by a factor of two. Hence, if L is of order 2p, then A_D is of order p, so that the requirements on the differentiability of the functions ϕ_j are lowered accordingly. More importantly, however, the energy integrals, Eq. (6), take automatically into account the natural boundary conditions, so that the functions ϕ_j need satisfy only the geometric boundary conditions. Such functions are sometimes referred to as energy functions. In more familiar terminology, we shall refer to functions belonging to H as comparison functions and those belonging to H_A as admissible functions (see Ref. 5).

3. Rayleigh's Quotient and the Rayleigh-Ritz Method

Let us assume that the interest lies in a self-adjoint positive definite continuous system whose eigenvalue problem is defined by Eqs. (1) and (2). Let v be a comparison function, satisfying all the boundary conditions (2), and write

$$\omega^2 = R(v) = \frac{\int_D vLv dD}{\int_D Mv^2 dD} \tag{7}$$

where R(v) is known as Rayleigh's quotient for continuous systems; it is always positive for positive definite systems. Then, defining a weighted norm $\|v\|_M^2 = (v,v)_M = \int_D Mv^2 dD$ and integrating the numerator of Eq. (7) by parts, Rayleigh's quotient (7) can be written in the equivalent form

$$\omega^2 = R(v) = \frac{[v,v]_A}{(v,v)_M} = \frac{|\!|\!| v |\!|\!|^2}{\|v\|_M^2} \tag{8}$$

The advantage of Rayleigh's quotient in the form of Eq. (8) over that in the form of Eq. (7) is that the function v in Eq. (8) need be only

an admissible function. In either form, it can be shown (Ref. 5) that Rayleigh's quotient has a stationary value in the neighborhood of any eigenfunction $u^{(r)}$. Moreover, the stationary value is a minimum in the neighborhood of the first eigenfunction, so that Rayleigh's quotient provides an upper bound for the first eigenvalue $\lambda_1 = \omega_1^2$ with respect to the class of either comparison or admissible functions, depending on whether Eq. (7) or Eq. (8) is used. The essence of the Rayleigh-Ritz method is to converge to the desired eigenvalues by constructing a minimizing sequence to represent v.

Let us consider an approximation to the solution of the eigenvalue problem (1) in the form of the finite series

$$v_n = \sum_{i=1}^{n} a_i \phi_i \tag{9}$$

where ϕ_i $(i=1,2,\ldots,n)$ are known functions of the spatial coordinates and a_i are unknown coefficients to be determined. The functions ϕ_i are linearly independent over the domain D and are either comparison functions or admissible functions depending on whether Rayleigh's quotient is in the form (7) or (8), respectively. Following a substitution of Eq. (9) into Rayleigh's quotient, the coefficients a_i are chosen so as to render the quotient stationary. It can be shown (Ref. 5) that, for Rayleigh's quotient to be stationary, the coefficients a_i must satisfy the n algebraic equations

$$\sum_{j=1}^{n} \{[\phi_i,\phi_j]_A - \Lambda^n(\phi_i,\phi_j)_M\}a_i = 0 \qquad 1 \;,\ldots,n \tag{10}$$

where the form (8) of Rayleigh's quotient has been used. Note that Λ^n denotes the estimated value of Rayleigh's quotient, in which n represents the number of terms in series (9). Equations (10) represent an algebraic eigenvalue problem for an n-degree-of-freedom discrete system. In fact,

the n-degree-of-freedom system can be regarded as the result of imposing the constraints

$$a_{n+1} = a_{n+2} = \dots = 0 \qquad (11)$$

on the infinitely-many-degree-of-freedom system. Because constraints have a tendency to raise the stiffness of the system, the estimated eigenvalues Λ_r^n (the square of the natural frequencies) will provide upper bounds for the true eigenvalues λ_r,

$$\Lambda_r^n \geq \lambda_r, \qquad r = 1,2,\dots,n \qquad (12)$$

The estimated eigenfunctions associated with the estimated eigenvalues Λ_r^n are obtained by introducing the coefficient $a_i^{(r)}$ in Eq. (9),

$$u_n^{(r)} = \sum_{i=1}^{n} a_i^{(r)} \phi_i \qquad (13)$$

The question remains as to how the estimated eigenvalues Λ_r^n and the coefficients $a_i^{(r)}$ behave in the limit. Be adding a term to the series (9), the number of constraints imposed on the system is reduced by one. If the eigenvalues Λ_r^n of the original system are such that $\Lambda_1^n \leq \Lambda_2^n \leq \dots \leq \Lambda_n^n$ and if the eigenvalues Λ_r^{n+1} of the system obtained by reducing the number of constraints by one are such that $\Lambda_1^{n+1} \leq \Lambda_2^{n+1} \leq \dots \leq \Lambda_{n+1}^{n+1}$, then by the inclusion principle (Ref. 3), we have

$$\Lambda_1^{n+1} \leq \Lambda_1^n \leq \Lambda_2^{n+1} \leq \dots \leq \Lambda_n^{n+1} \leq \Lambda_n^n \leq \Lambda_{n+1}^{n+1} \qquad (14)$$

The inclusion principle permits us to conclude that the estimated eigenvalues tend to decrease with each additional degree of freedom. At the same time there is a new eigenvalue added which is higher than any of the previous ones. The estimated eigenvalues decrease monotonically, as the number of degrees of freedom increases. Hence, as $n\to\infty$ the estimated eigenvalues Λ_r^n approach the true eigenvalues from above,

$$\lim_{n\to\infty} \Lambda_r^n = \lambda_r \qquad r = 1,2,\dots \qquad (15)$$

The question of convergence of the coefficients $a_i^{(r)}$ depends on the set of functions ϕ_i. Determining the coefficients $a_i^{(r)}$ as solutions to the eigenvalue problem (10) is equivalent to finding the best approximation to the true eigenfunctions $u^{(r)}$ *with respect to the energy norm*, i.e.,

$$\| u^{(r)} - u_n^{(r)} \|_A = \| u^{(r)} - \sum_{i=1}^{n} a_i^{(r)} \phi_i \|_A = \text{minimum} \qquad (16)$$

Hence, we conclude that if the set of functions ϕ_i is complete in the energy space H_A, $u_n^{(r)}$ converges in energy to $u^{(r)}$. Furthermore, if the set of functions ϕ_i is minimal* in the energy space H_A, then limits for the coefficients ${}^n a_i^{(r)}$ exist and have the values

$$a_i^{(r)} = \lim_{n\to\infty} {}^n a_i^{(r)} \qquad i = 1,2,... \qquad (17)$$

Moreover, if the system of functions which is orthonormal to the set ϕ_i is bounded in H_A, then the limiting process in Eq. (17) is uniform with respect to i (Ref. 6).

Implicit in the above discussion is the need to approximate a continuous system by a discrete one. Indeed, the process of constructing the series (9) is called discretization because it reduces the problem of finding an infinite set of continuous functions u to that of determining a finite number of constant coefficients a_i. Of course, the object is to obtain an accurate approximation to the lowest eigenvalues and eigenfunctions while taking only a small number of n functions in Eq. (9). The question remains as to how to select the admissible functions ϕ_i in Eq. (9).

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* A system of elements of the Hilbert space H_A is called a minimal system in H_A if the deletion of any one of the elements from the system restricts the span of this new set to a proper subspace of the space spanned by the original system (Ref. 6).

4. On the Selection of Admissible Functions

An in-depth discussion on the rational choice of the functions ϕ_i can be found in Ref. 6. The discussion in Ref. 6 is concerned with many aspects of the problem of selection of the functions ϕ_i, including the numerical stability of the approximate solution. However, problems of numerical stability are of no concern here, because quite often in engineering it is possible to obtain an accurate approximate solution of the eigenvalue problem before numerical stability becomes a problem. As a result, we will be concerned only with obtaining a set of suitable admissible functions. Hence, ignoring numerical stability aspects, criteria for the selection of the functions ϕ_i in a closed domain $\bar{D}$ are:

1) Any finite set of admissible functions must be linearly independent.

2) The set of admissible functions must be complete in the energy space H_A of the domain $\bar{D}$.

In addition, as mentioned in Sec. 3, limits for the coefficients a_i in the series (9) exist if the set of admissible functions ϕ_i is minimal.

The second of these criteria requires that we be able to identify a set of admissible functions that is complete in the energy space H_A. To this end, the following statement is useful: If A and B are positive definite operators and the spaces H_A and H_B contain the same elements, then any set that is complete in H_B is complete in H_A. Therefore, by letting B have a simple form for which a complete set of eigenfunctions is easily found, we can use that set as a set of admissible functions for the operator A.

The above discussion says nothing about the rate of convergence of approximate eigenvalues and eigenfunctions for a particular set of functions satisfying the above criteria. In fact, the approximate eigensolution will not necessarily converge very rapidly to the exact solution. However, in practice, we are concerned with finding only the lower eigenvalues and eigenfunctions and relatively simple admissible functions are known to yield rapid convergence to the lower eigenvalues and eigenfunctions for many problems. Therefore, the lack of analytic knowledge about the rate of convergence of the approximate eigensolution for a particular set of admissible functions is not a deterrent to using the Rayleigh-Ritz method. Indeed, experience has shown that, by comparing the approximate lower eigenvalues and eigenfunctions obtained by using several different sets of admissible functions, one can find a set of functions which yields good convergence.

So far we have been concerned with selecting a set of admissible functions, i.e., a set of functions which are p times differentiable and satisfy the geometric boundary conditions for the eigenvalue problem (1). We shall now consider a method of obtaining not just admissible functions but comparison function, i.e., functions which are 2p times differentiable and satisfy both the geometric and natural boundary conditions of the eigenvalue problem (1). The method is based on the integral formulation of the eigenvalue problem (1). We point out that, while the method is not practical for finding comparison functions, an analogous method for the discrete problem will prove very useful. Indeed, the motivation for the present discussion is the extension to the discrete problem.

The eigenvalue problem (1) can be written in the equivalent integral form (see Ref. 5)

$$u(P) = \lambda \int_D G(P,Q)M(Q)u(Q)dD(Q) \tag{18}$$

where P denotes the position of the point at which the displacement $u(P)$ is measured and $G(P,Q)$ is an influence function, known as a Green's function. The Green's function $G(P,Q)$ satisfies all the boundary conditions of the problem. Moreover, for a self-adjoint system, the Green's function is symmetric in P and Q, $G(P,Q) = G(Q,P)$. Equation (18) can be solved by iteration. Denoting by $u_0(P)$ the first trial function, the first iterated function $u_1(P)$ is given by

$$u_1(P) = \int_D G(P,Q)M(Q)u_0(Q)dD(Q) \tag{19}$$

Then, introducing $u_1(Q)$ in the integrand, we obtain $u_2(P)$, etc. We are not concerned here with iterating to the actual first mode but with the properties of the functions $u_0(P)$ and $u_1(P)$. Indeed, if $u_0(P)$ is any function which is continuous in the domain D, then $u_1(P)$ will be 2p times differentiable and satisfy all the boundary conditions, that is $u_1(P)$ will be a comparison function. In addition, for a self-adjoint positive definite system, if $u_0^i(P)$ $(i=1,2,...)$ is a set of linearly independent continuous functions, then the set of functions $u_1^i(P)$ $(i=1,2,...)$ will also be linearly independent.

Let us consider as an example the bending of a uniform cantilevered beam clamped at x=0. In this case Green's function $G(P,Q)$ reduces to the flexibility influence function $a(x,\xi)$ where

$$a(x,\xi) = \begin{cases} \frac{1}{EI}\frac{\xi^2}{2}(x - \frac{1}{3}\xi), & \xi \le x \\ \frac{1}{EI}\frac{x^2}{2}(\xi - \frac{1}{3}x), & \xi > x \end{cases} \tag{20}$$

in which EI is the bending stiffness of the beam. The eigenvalue problem has the form

$$EI\, u''''(x) = \lambda m u(x), \quad 0 < x < L \tag{21}$$

where $u''''(x) = d^4u(x)/dx^4$ and m is the mass per unit length. In Eq. (21) $u(x)$ satisfies the four boundary conditions

$$u(0) = u'(0) = u''(L) = u'''(L) = 0 \tag{22}$$

Let us consider the two linearly independent trial functions

$$u_0^1(x) = C \tag{23a}$$

$$u_0^2(x) = \frac{x}{L} \tag{23b}$$

Clearly, these functions not only are not comparison functions but they are not even admissible functions, as they violate the geometric boundary conditions at $x=0$. Then,

$$u_1^i(x) = \int_0^L a(x,\xi) m\, u_0^i(\xi) d\xi, \quad i = 1,2 \tag{24}$$

where

$$u_1^1(x) = \frac{CmL^4}{4EI}\left(\frac{x^2}{L^2} - \frac{2}{3}\frac{x^3}{L^3} + \frac{1}{6}\frac{x^4}{L^4}\right)$$

$$u_1^2(x) = \frac{mL^4}{6EI}\left(\frac{x^2}{L^2} - \frac{1}{2}\frac{x^3}{L^3} + \frac{1}{20}\frac{x^5}{L^5}\right) \tag{25}$$

and it is easy to verify that $u_1^1(x)$ and $u_1^2(x)$ satisfy the four boundary conditions (22) and are linearly independent.

5. The Discrete Problem

The eigenvalue problem (10) represents a discrete approximation to the continuous eigenvalue problem defined by Eqs. (1) and (2). Quite often, however, engineering problems are posed directly in terms of an n degree of freedom discrete system yielding the eigenvalue problem

$$K\underset{\sim}{u} = \lambda M\underset{\sim}{u} \tag{26}$$

where K and M are $n \times n$ positive definite symmetric stiffness and mass matrices, respectively, and $\underset{\sim}{u}$ is an n-dimensional configuration vector.

If the system of Eq. (26) represents a mathematical model of an actual structure, then the number of degrees of freedom may be in the hundreds or even thousands. Quite often, it is not feasible to work with such large systems and the object is to reduce the number of degrees of freedom while retaining the essential dynamic characteristics of the original system. The approach we shall consider parallels the method presented previously for continuous systems. But first we shall present the mathematical concepts for discrete systems by constructing an analogy with the concepts presented in Secs. 2-4 for continuous systems.

Let H be an n-dimensional vector space and consider the n-dimensional vector $\underset{\sim}{u}$ in H. We define the norm $\|\underset{\sim}{u}\|$ of $\underset{\sim}{u}$ by $\|\underset{\sim}{u}\|^2 = (\underset{\sim}{u},\underset{\sim}{u}) = \underset{\sim}{u}^T\underset{\sim}{u}$. If we wish to approximate $\underset{\sim}{u}$ by some sequence of vectors $\underset{\sim}{u}_1$, $\underset{\sim}{u}_2$, ..., $\underset{\sim}{u}_N$ $(N \leq n)$ in H, and if $\lim_{N\to n} \|\underset{\sim}{u}_N - \underset{\sim}{u}\| = 0$, then we say that $\underset{\sim}{u}_N$ converges in the mean to $\underset{\sim}{u}$. Let us assume that the vector $\underset{\sim}{u}$ is represented by the series $\underset{\sim}{u} = \sum_{j=1}^{n} a_j \underset{\sim}{\phi}_j$ which is convergent in the mean, and consider the partial sum

$$\underset{\sim}{u}_N = \sum_{j=1}^{N} a_j \underset{\sim}{\phi}_j , \qquad N \leq n \tag{27}$$

In addition, we shall define the energy norm for discrete systems by $|\!|\!|\underset{\sim}{u}|\!|\!|^2 = [\underset{\sim}{u},\underset{\sim}{u}] = \underset{\sim}{u}^T K \underset{\sim}{u}$. The sequence of vectors $\underset{\sim}{u}_1$, $\underset{\sim}{u}_2$, ..., $\underset{\sim}{u}_N$ converges in energy to $\underset{\sim}{u}$ if $\lim_{N\to n} |\!|\!|\underset{\sim}{u}_N - \underset{\sim}{u}|\!|\!| = 0$. If $\underset{\sim}{u}_N$ represents the partial sum (27) then the linearly independent set of vectors $\underset{\sim}{\phi}_j$ is also complete in energy. Of course, for finite-dimensional spaces it is not hard to see that the energy space H_A is the same as the n-dimensional vector space H. If we now define the weighted norm $\|\underset{\sim}{u}\|_M^2 = (\underset{\sim}{u},\underset{\sim}{u})_M = \underset{\sim}{u}^T M \underset{\sim}{u}$, we can write Rayleigh's quotient for the discrete system (26) as

$$\omega^2 = R(\underset{\sim}{v}) = \frac{[\underset{\sim}{v},\underset{\sim}{v}]}{(\underset{\sim}{v},\underset{\sim}{v})_M} = \frac{\|\underset{\sim}{v}\|^2}{\|\underset{\sim}{v}\|_M^2} \tag{28}$$

where $\underset{\sim}{v}$ is an arbitrary n-dimensional vector. The Rayleigh quotient (28) has a stationary value when the vector $\underset{\sim}{v}$ is in the neighborhood of any eigenvector $\underset{\sim}{u}^{(r)}$ $(r=1,2,\ldots,n)$ and the stationary value is a minimum in the neighborhood of the first eigenvector.

Next, invoking the analogy with the continuous system, we select a set of trial vectors $\underset{\sim}{\phi}_i$ and construct a linear combination

$$\underset{\sim}{v}_N = \sum_{i=1}^{N} a_i \underset{\sim}{\phi}_i \quad (N \leq n) \tag{29}$$

where the $\underset{\sim}{\phi}_i$ are known vectors and the a_i are undetermined coefficients. Substituting Eq. (29) into Eq. (28), we choose the a_i so as to render the quotient (28) stationary, yielding the N algebraic equations

$$\sum_{j=1}^{N} \{[\underset{\sim}{\phi}_i, \underset{\sim}{\phi}_j] - \Lambda^N (\underset{\sim}{\phi}_i, \underset{\sim}{\phi}_j)_M\} a_i = 0 \tag{30}$$

Equations (30) represent an N-dimensional eigenvalue problem, so that the dimension of the eigenvalue problem (26) has been reduced from n to N. As before, the estimated eigenvalues Λ_r^N provide upper bounds for the true eigenvalues λ_r, $\Lambda_r^N \geq \lambda_r$, and the estimated eigenvectors are obtained by introducing the coefficients $a_i^{(r)}$ into Eq. (29)

$$\underset{\sim}{u}_N^{(r)} = \sum_{i=1}^{N} a_i^{(r)} \underset{\sim}{\phi}_i \tag{31}$$

The limit properties of the estimated eigenvalues Λ_r^N and the coefficients $a_i^{(r)}$ for the discrete system are very simple. This is because any set of n linearly independent vectors $\underset{\sim}{\phi}_i$ is both minimal in the energy space and complete in energy. Hence, if the vectors $\underset{\sim}{\phi}_i$ are all linearly independent, then mathematically we have

$$\lim_{N \to n} \Lambda_r^N = \lambda_r, \quad \lim_{N \to n} \| \underset{\sim}{u}^{(r)} - u_N^{(r)} \| = 0$$

$$\lim_{N \to n} {}^N a_i^{(r)} = a_i^{(r)} \tag{32}$$

and the limiting process for the coefficients $a_i^{(r)}$ is automatically uniform, i.e., all the limiting values are reached when N=n.

The question remains as to how to choose the vectors $\underset{\sim}{\phi}_i$. It is obvious that for discrete systems the mathematical analysis does not impose restrictions on the vectors $\underset{\sim}{\phi}_i$ other than linear independence. In other words, the analysis does not guarantee the satisfaction of the boundary conditions and smoothness requirements for the discrete vectors $\underset{\sim}{\phi}_i$, so that these features must be built into the vectors $\underset{\sim}{\phi}_i$ in other ways. In fact, if the discrete system represents the mathematical model for a continuous structure the boundary conditions and smoothness requirements are already taken into account in the discrete model. This suggests that the smoothness properties of the admissible vectors can be obtained from the discrete model, a subject discussed in the next section. Our interest is in selecting a small set of N vectors $\underset{\sim}{\phi}_i$ which will yield accurate results for the lower eigenvalues and eigenvectors of the n-dimensional discrete system.

6. The Concept of Admissible Vectors

It is here postulated that the concept of admissible functions, encountered in conjunction with the mathematical analysis of continuous systems, can be extended to discrete systems, giving rise to what will be referred to as "admissible vectors". Essentially, the idea is that for a particular discrete system one can conceive of a simple continuous system that is similar dynamically to the discrete system. Such a

continuous system admits a set of admissible functions $\phi_i(D)$. By choosing an appropriate set of n discrete points in the domain D of the continuous system, we can construct a set of n-dimensional admissible vectors $\underset{\sim}{\phi}_i$ with the n entries corresponding to the values of the functions $\phi_i(D)$ at the n discrete points. This process guarantees that the admissible vectors are in some way "smooth". Clearly, if the admissible functions are linearly independent, then the admissible vectors constructed by "discretizing" these functions will also be linearly independent. Moreover, there is the possibility that other properties of admissible functions carry over to admissible vectors. The validity of the concept can be seen by considering the following example.

Consider the discrete system consisting of 12 identical masses connected to a support and to each other by 12 identical flexible massless links with bending stiffness EI (see Fig. 1). The support is assumed to rotate with the uniform angular velocity Ω. We shall consider only the displacement of each mass from the plane of rotation. Taking each discrete mass to be equal to m, the 12×12 mass matrix M is simply the 12×12 identity matrix multiplied by m. Calculation of the stiffness matrix K is quite a bit more involved. In fact, it will prove convenient not to calculate K itself but to calculate its inverse K^{-1}, where $K^{-1} = A$ is recognized as the system flexibility matrix. The entries a_{ij} of the matrix A are known as the flexibility influence coefficients, where the latter are the discrete counterparts of Green's function. Obtaining the influence coefficients a_{ij} is no simple task in the case at hand. Because the support rotates with the uniform angular velocity Ω, the coefficient a_{ij} will depend on all the coefficients

a_{kj} (k=1,2,...,12). Hence, for each j (j=1,2,...,12) the coefficients a_{kj} (k=1,2,...,12) can be shown to be the solutions to the 12 simultaneous algebraic equations

$$\frac{\Omega^2 h^3 m}{2EI} \sum_{R=1}^{i-1} a_{\ell j} \{\frac{1}{3}\ell(3i-1) + \ell[\sum_{r=1}^{\ell-1} 2(i-r)] - 2(i-\ell)(\sum_{k=\ell+1}^{12} k)\}$$

$$+ a_{ij} \{1 + \frac{\Omega^2 h^3 m}{2EI} [\frac{i}{3} (3i-1) + 2i \sum_{r=1}^{i-1} (i-r) - \frac{1}{3} (\sum_{k=i+1}^{12} k)]\}$$

$$+ \frac{\Omega^2 h^3 m}{2EI} (\sum_{k=i+1}^{12} k a_{kj})[i + 2 \sum_{r=1}^{i-1} (i-r)] = \frac{1}{6} \frac{h^3}{EI} j^2(3i-j)$$

$$i = 1,2,\ldots,12 \qquad (33)$$

where the distance between each mass is assumed to be h. Note that for each j (j=1,2,...,12) we must solve a system of 12 equations in the 12 unknowns a_{kj} (k=1,2,...,12). The coefficients a_{ij} were found numerically taking h=L/12 and $\Omega^2 = (1.2)^4 \frac{EI}{m}$ and are exhibited in Table 1. Hence, we are concerned with the discrete eigenvalue problem

$$\lambda m A \underset{\sim}{u} = \underset{\sim}{u} \qquad (34)$$

where $\underset{\sim}{u}$ is a 12-dimensional vector.

The eigenvalue problem (34) can be solved numerically for the 12 actual eigenvalues λ_r and associated eigenvectors $\underset{\sim}{u}_r$. We wish to compare the actual eigensolution of the 12th-order eigenvalue problem (34) with the solution of the reduced eigenvalue problem (30), which in terms of the flexibility matrix A takes the form

$$\sum_{j=1}^{N} \{\Lambda^N m \underset{\sim}{\phi}_i^T A \underset{\sim}{\phi}_j - \underset{\sim}{\phi}_i^T \underset{\sim}{\phi}_j\} a_i = 0 \qquad , \quad i = 1,2,\ldots,N \qquad (35)$$

The eigenvalue problem (35) has been solved five times using different sets of vectors $\underset{\sim}{\phi}_i$. The first solution was obtained by using an expansion in terms of the eigenvectors of the nonrotating discrete system.

The first five eigenvalues and eigenvectors of the nonrotating discrete system are displayed in Table 2. The eigenvalues of the reduced-order rotating system using a two-term, three-term, four-term and five-term expansion in terms of the nonrotating eigenvectors are displayed in Column a of Table 3. Note that the eigenvectors of the nonrotating discrete system possess all the smoothness properties of the discrete system. A set of vectors possessing these smoothness properties is what we will call a set of admissible vectors. Indeed, the bending of a cantilevered beam represents a continuous system which is similar to the present discrete system so that the smoothness properties of the set of admissible vectors correspond to the differentiability and boundary conditions requirements imposed on the set of admissible functions of this similar continuous system. The admissible functions for a cantilevered beam are functions which are twice differentiable and have the displacement and slope at the origin equal to zero.

The second solution of the reduced order discrete system was obtained by using an expansion in terms of "discretized" admissible functions. We take as a set of admissible functions of the cantilevered beam the functions

$$\phi_r(x) = (x/L)^{r+1} \qquad r = 1,2,\ldots \tag{36}$$

and construct the vectors $\underset{\sim}{\phi}_r$ by taking as the i^{th} entry in the 12-dimensional vector $\underset{\sim}{\phi}_r$ the value of the function ϕ_r at $\frac{x}{L} = ih$ $(i=1,2,\ldots,12)$. Therefore,

$$\underset{\sim}{\phi}_r = \{(h)^{r+1}, (2h)^{r+1}, \ldots, (12h)^{r+1}\}^T \tag{37}$$

The eigenvalues of the reduced-order system using the vectors of Eq. (37) are displayed in Column b of Table 3 for r=2,3,4,5, respectively. Of course, it is easily verified that the functions (36) are twice differentiable and that $\phi_r(0) = \phi_r'(0) = 0$. Hence, the vectors (37)

possess the smoothness of the functions ϕ_r.

The third solution of the reduced discrete system was obtained by taking discrete values of the functions

$$\phi_r(x) = 1 - \cos(r\pi x/L) + \frac{1}{2}(-1)^{r+1}(r\pi x/L)^2 \tag{38}$$

at the points $\frac{x}{L} = ih$ $(i=1,2,\ldots,12)$ as the entries of the vectors $\underset{\sim}{\phi}_r$. The eigenvalues are displayed in Column c of Table 3 for r=2,3,4,5, respectively. Note that the functions (38) are not only admissible functions but comparison functions, i.e. they are four times differentiable and satisfy all the boundary conditions $\phi_r(0) = \phi_r'(0) = \phi_r''(L) = \phi_r'''(L) = 0$.

For comparison purposes, the fourth solution of the reduced discrete system was obtained by using a set of vectors which possess none of the smoothness characteristics of the system. The motivation is the following: if we use a complete set of twelve arbitrary linearly independent vectors, then we obtain the actual solution only in terms of a different coordinate system. Hence, the question arises as to what kind of results will be obtained by using a reduced set of these vectors. The following vectors have been used:

$$\begin{aligned}
\underset{\sim}{\phi}_1 &= \{1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\}^T \\
\underset{\sim}{\phi}_2 &= \{1\ 1\ 1\ 1\ 1\ 1\ {-1}\ {-1}\ {-1}\ {-1}\ {-1}\ {-1}\}^T \\
\underset{\sim}{\phi}_3 &= \{1\ 1\ 1\ {-1}\ {-1}\ {-1}\ {-1}\ {-1}\ {-1}\ 1\ 1\ 1\}^T \\
\underset{\sim}{\phi}_4 &= \{1\ 1\ 1\ {-1}\ {-1}\ {-1}\ 1\ 1\ 1\ {-1}\ {-1}\ {-1}\}^T \\
\underset{\sim}{\phi}_5 &= \{1\ 1\ {-1}\ {-1}\ {-1}\ 1\ 1\ {-1}\ {-1}\ {-1}\ 1\ 1\}^T
\end{aligned} \tag{39}$$

The vectors (39) are not smooth in any sense. The eigenvalues of the reduced system using a two-, three-, four-, and five-term expansion in terms of the vectors (39) are displayed in Column d of Table 3.

Next, let us consider the discrete analog of the integral formulation introduced in Sec. 4. The matrix form analogous to Eq. (19) is

$$\underset{\sim}{u}_1 = K^{-1}M\underset{\sim}{u}_0 = AM\underset{\sim}{u}_0 \tag{40}$$

As discussed in Ref. 7, Eq. (40) is the basis for an iterative procedure to obtain the actual eigenvectors. However, because our interest is merely in selecting a set of vectors which will yield a good representation of the system dynamic characteristics, it is not necessary for us to iterate to the actual eigenvectors. Indeed, we shall use Eq. (40) only as a way of imposing the smoothness properties of the discrete system on a set of arbitrary linearly independent vectors. If the system is positive definite then the resulting vectors will also be linearly independent. Because the matrix A has built into it the system smoothness characteristics and the equivalent of boundary conditions, by analogy with the continuous case, the vector $\underset{\sim}{u}_1$ will possess all the system smoothness characteristics regardless of the form of $\underset{\sim}{u}_0$. Hence, the fifth solution of the reduced discrete system was obtained by using the vectors (39) as trial vectors on the right side of Eq. (40) and regarding the resulting vectors as admissible vectors. The eigenvalues for a two-, three-, four-, and five-term expansion are displayed in Column e of Table 3.

The actual eigenvalues of the 12 by 12 system are displayed in Column f of Table 3. To compare the actual system eigensolution with the eigensolutions of the reduced order problem (35) for the five different sets of admissible vectors just discussed, the error in the approximate eigenvalues as a percentage of the actual eigenvalues

$$\varepsilon_i^N = \left(\frac{\Lambda_i^N - \lambda_i}{\lambda_i} \right) \times 100 \tag{41}$$

has been calculated for two, three, four and five term expansions for each set of admissible vectors. The results are displayed in Table 4,

where Column a of Table 4 gives the percent error in the eigenvalues of Column a of Table 3, and similar statements hold for Columns b, c, d and e. In addition, the actual system eigenvectors $\tilde{u}^{(r)}$ as well as the approximate eigenvectors $\tilde{u}_N^{(r)}$ obtained from the various reduced order eigenvalue problems have been calculated. The square root of the norms of the errors of the approximate eigenvectors

$$\delta_i^N = \| \tilde{u}_N^{(i)} - \tilde{u}^{(i)} \|^{1/2} \tag{42}$$

are displayed in Table 5 for two, three, four and five term expansions, where Column a of Table 5 contains the norms of the errors of the eigenvectors associated with the eigenvalues displayed in Column a of Table 3 and similar correspondences hold for Columns b, c, d, and e. Tables 4 and 5 permit us to make several remarks about the relationship between the accuracy of the approximate eigensolutions and the set of admissible vectors used. First, Column a of Tables 4 and 5 reveals that in using the eigenvectors of the nonrotating discrete system to reduce the order of the rotating discrete system one obtains approximate lower eigenvalues and eigenvectors which differ only slightly from the actual lower eigenvalues and eigenvectors. This can be traced to the similarity of the eigenvectors of the nonrotating discrete system with the eigenvectors of the rotating discrete system. Hence, the choice of the eigenvectors of the nonrotating discrete system as admissible vectors is a good choice provided the eigenvalue problem for the nonrotating discrete system can be easily defined and solved. However, if only the lower eigenvalues and eigenvectors of the rotating system are of interest, other sets of admissible vectors yield equally good results as can be seen by inspection of Columns b, c and e of Tables 4 and 5. Note that using the truncated set of linearly independent vectors (39) as admissible vectors

yields very poor results while using as admissible vectors those vectors obtained by imposing the system smoothness characteristics on the vectors (39) via Eq. (40) yields very good results. This can be seen by inspection of Columns d and e of Tables 4 and 5. Moreover, a comparison of Columns b and e of Tables 4 and 5 suggests that the error in the approximate eigensolution obtained by using "discretized" admissible vectors is similar to the error in the approximate eigensolution obtained by using the vectors (39) "smoothed" by Eq. (40).

The degree of accuracy of the approximate eigenvectors can also be seen by plotting the normalized approximate eigenvectors $\underset{\sim}{u}_N$. Plots of these eigenvectors for two, three, four and five term expansions are shown in Figures 2, 3, 4 and 5, respectively. The approximate eigenvectors are obtained by using admissible vectors as in Columns a, b, c and e of Tables 4 and 5 and are plotted on the same axes as the actual eigenvectors. It is apparent from Figs. 2-5 that the first eigenvector is always very accurate, regardless of the choice and the number of admissible vectors used. Moreover, as we increase the number of admissible vectors used in the expansion, the accuracy of the second and third approximate eigenvectors improves regardless of the choice of admissible vectors. In short, Columns a, b, c and e of Tables 4 and 5 along with Figs. 2-5 permit us to conclude that any set of admissible vectors, i.e. vectors possessing the system smoothness characteristics, will yield accurate values for the lower eigenvalues and associated eigenvectors. This conclusion should have far-reaching implications in the dynamic simulation of complex structures by the substructure synthesis.

7. Summary and Conclusions

In this investigation, the concept of "admissible vectors" for

the simulation of discrete structures is advanced and placed on a sound mathematical foundation. The idea is to simulate the motion by a subspace of smaller dimnesion than the dimension of the actual space, i.e., to truncate. Admissible functions have been shown to be sufficient for the simulation of distributed-parameter members in a structure consisting of a given number of substructures. Admissible vectors are intended to play the same role for discrete substructures. The implication is that a set of vectors satisfying certain smoothness conditions and boundary conditions will yield a satisfactory representation of the substructure motion. Methods for generating admissible vectors are also presented. To introduce and validate the concept of admissible vectors, a discrete system in the form of a lumped-parameter rotating helicopter blade has been considered. The lower eigenvalues and eigenvectors obtained by describing the system in terms of different sets of admissible vectors have been shown to converge rapidly to the actual eigenvalues and eigenvectors. These preliminary results appear most promising.

8. References

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Table 1 - Matrix of Flexibility Influence Coefficients (x 10^3)

.1562296	.3457976	.4885085	.5974126	.6820749	.7495188	.8049268	.8521473	.8940529	.9327795	.9698719	1.006356
.3457976	1.024016	1.608339	2.054242	2.400888	2.677034	2.903899	3.097242	3.268822	3.427386	3.579260	3.728642
.4885085	1.608339	2.876412	3.918814	4.729179	5.374735	5.905086	6.357068	6.758178	7.128858	7.483897	7.833113
.5974126	2.054242	3.918814	5.773440	7.291306	8.500474	9.493855	10.34045	11.09175	11.78606	12.45107	13.10517
.6820749	2.400888	4.729179	7.291306	9.724155	11.74012	13.39631	14.80777	16.06037	17.21794	18.32667	19.41720
.7495188	2.677034	5.374735	8.500474	11.74012	14.77443	17.34750	19.54035	21.48639	23.28479	25.00731	26.70157
.8049268	2.903899	5.905086	9.493855	13.39631	17.34750	21.06330	24.31317	27.19726	29.86254	32.41537	34.92632
.8521473	3.097242	6.357068	10.34045	14.80777	19.54035	24.31317	28.87037	33.00109	36.81844	40.47471	44.07101
.8940529	3.268822	6.758178	11.09175	16.06037	21.48639	27.19726	33.00109	38.66401	43.98721	49.08580	54.10075
.9327795	3.427386	7.128858	11.78606	17.21794	23.28479	29.86254	36.81844	43.98721	51.14785	58.09933	64.93679
.9698719	3.579260	7.483897	12.45107	18.32667	25.00731	32.41537	40.47471	49.08580	58.09933	67.28918	76.42359
1.006356	3.728642	7.833113	13.10517	19.41720	26.70157	34.92632	44.07101	54.10075	64.93679	76.42359	88.29382

Table 2

Eigenvalues and Eigenvectors of the Nonrotating Discrete System

$$\Lambda = \text{diag}\,[.8776586 \quad 34.71710 \quad 273.9302 \quad 1058.188 \quad 2904.5644]$$

$$U = \begin{bmatrix}
.6144262\cdot10^{-2} & .3506042\cdot10^{-1} & .8938545\cdot10^{-1} & .1574684 & .2313173 \\
.2363973\cdot10^{-1} & .1198151 & .2648566 & .3836237 & .4309673 \\
.5108426\cdot10^{-1} & .2244364 & .4026492 & .4044953 & .2050418 \\
.8708854\cdot10^{-1} & .3215703 & .4199895 & .1600834 & -.2351863 \\
.1302899 & .3883468 & .2947513 & -.1889837 & -.3938290 \\
.1793706 & .4082550 & .6761981\cdot10^{-1} & -.3947276 & -.9143334\cdot10^{-1} \\
.2330795 & .3724292 & -.1766866 & -.3102714 & .3186319 \\
.2902570 & .2800194 & -.3431838 & -.8415515\cdot10^{-3} & .3594727 \\
.3498619 & .1374686 & -.3628411 & .3012934 & -.1365950\cdot10^{-1} \\
.4110011 & -.4332252\cdot10^{-1} & -.2171513 & .3626053 & -.3572553 \\
.4729589 & -.2478191 & .5924247\cdot10^{-1} & .1094921 & -.2424909 \\
.5352285 & -.4624363 & .4007263 & -.3416881 & .2873619
\end{bmatrix}$$

Table 3 - Discrete System Eigenvalues

No. of Terms	Estimated Eigenvalues	a	b	c	d	e	f
2	$\lambda_1 \times 10^{-1}$	.3289033	.3289754	.3289252	.3566144	.3288961	.3288935
	$\lambda_2 \times 10^{-2}$	.4804341	.5075637	.4811161	1.133973	.4929387	.4804187
3	$\lambda_1 \times 10^{-1}$	.3288940	.3288953	.3288962	.3559324	.3288937	.3288935
	$\lambda_2 \times 10^{-2}$	.4804338	.4804287	.4808920	.5999323	.4805295	.4804187
	$\lambda_3 \times 10^{-3}$	.3099567	.3498520	.3111606	.5434759	.3166433	.3099386
4	$\lambda_1 \times 10^{-1}$	.3288936	.3288935	.3288940	.3349127	.3288935	.3288935
	$\lambda_2 \times 10^{-2}$	.4804197	.4804286	.4804447	.5578686	.4804286	.4804187
	$\lambda_3 \times 10^{-3}$	.3099452	.3105774	.3111565	.4531640	.3109182	.3099386
	$\lambda_4 \times 10^{-4}$	.1129088	.1361532	.1139375	.2147814	.1286027	.1129031
5	$\lambda_1 \times 10^{-1}$	.3288935	.3288935	.3288936	.3348560	.3288935	.3288935
	$\lambda_2 \times 10^{-2}$	.4804189	.4804192	.4804359	.5514864	.4804219	.4804187
	$\lambda_3 \times 10^{-3}$	.3099391	.3101618	.3100193	.4527830	.3102866	.3099386
	$\lambda_4 \times 10^{-4}$	.1129044	.1141827	.1138691	.1563050	.1133845	.1129031
	$\lambda_5 \times 10^{-5}$	.3019821	.3855055	.3066433	.4084688	.3156557	.3019716

a. Nonrotating eigenvectors as admissible vectors

b. Admissible vectors as per Eq. (37) ($\phi_{ji} = (jh)^{i+1}$)

c. Admissible vectors as per Eq. (38) ($\phi_{ji} = 1 - \cos jh\pi i + \frac{1}{2} (-1)^{i+1} (jh\pi i)^2$)

d. Admissible vectors as per Eq. (39)

e. Admissible vectors as per Eq. (40)

f. Actual Eigenvalues

Table 4 - Relative Error in the Estimated Eigenvalues (in %)

No. of Terms	Relative Error	a	b	c	d	e
2	ε_1	0.00296711	0.02488588	0.00962596	8.42853092	0.00077392
	ε_2	0.00320669	5.65029065	0.14517522	136.03847919	2.60607359
3	ε_1	0.00014806	0.00053885	0.00081243	8.22115504	0.00004408
	ε_2	0.00314882	0.00209765	0.09851756	24.87698487	0.02307392
	ε_3	0.00584250	12.87784569	0.39425222	75.34951656	2.16321827
4	ε_1	0.00001622	0.00001260	0.00016299	1.83013831	0.00000064
	ε_2	0.00020903	0.00206151	0.00542953	16.12134874	0.00207058
	ε_3	0.00211595	0.20608765	0.39295927	46.21087947	0.31604395
	ε_4	0.00503804	20.59293695	0.91621720	90.23517802	13.90534287
5	ε_1	0.00000258	0.00000009	0.00003613	1.81288300	0.00000017
	ε_2	0.00004674	0.00011852	0.00358253	14.79286722	0.00068427
	ε_3	0.00016512	0.07201388	0.02603060	46.08796571	0.11228190
	ε_4	0.00114977	1.13339044	0.85563648	38.44174387	0.42634579
	ε_5	0.00349043	27.66285680	1.54706828	35.26728795	4.53158209

Note: Values in Columns a, b, ..., e correspond to admissible vectors as in Table 3.

Table 5 - The Norm of the Error in the Eigenvectors

No. of Terms	Norm of the Error	a	b	c	d	e
2	δ_1	0.00547506	0.01588587	0.00986074	0.28397162	0.00279942
	δ_2	0.00578902	0.25104090	0.03974361	0.90269989	0.17395070
3	δ_1	0.00121843	0.00232567	0.00285403	0.27983097	0.00066492
	δ_2	0.00572897	0.00461749	0.03204797	0.47120213	0.01554358
	δ_3	0.00865562	0.38862998	0.06618765	0.76453658	0.17048115
4	δ_1	0.00040298	0.00035519	0.00127729	0.13463032	0.00008023
	δ_2	0.00145583	0.00458223	0.00740622	0.38371777	0.00459591
	δ_3	0.00484279	0.04851585	0.06602919	0.61530317	0.06034538
	δ_4	0.00871355	0.49721320	0.10629355	0.94064578	0.45201685
5	δ_1	0.00016066	0.00002935	0.00060122	0.13397904	0.00004093
	δ_2	0.00068589	0.00109361	0.00600685	0.36878886	0.00262626
	δ_3	0.00130988	0.02765621	0.01633764	0.61246579	0.03441519
	δ_4	0.00370327	0.11857407	0.10113619	0.60093044	0.07256947
	δ_5	0.00778591	0.58238777	0.14596192	0.68919375	0.28400738

Note: Values in Columns a, b, ..., e correspond to admissible vectors as in Table 3.

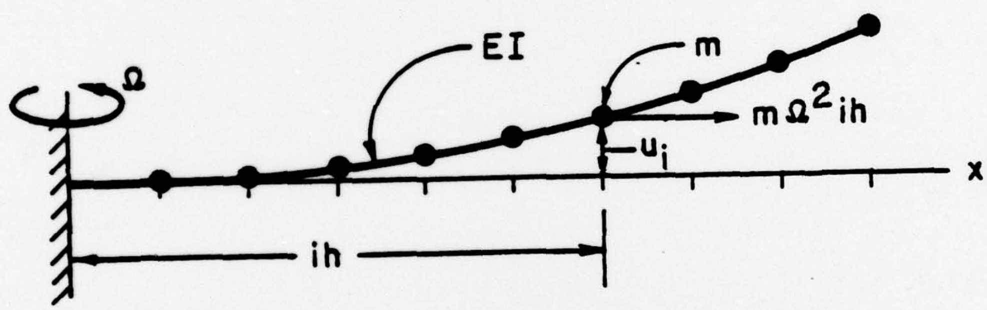


FIGURE 1 - DISCRETE SYSTEM MODEL

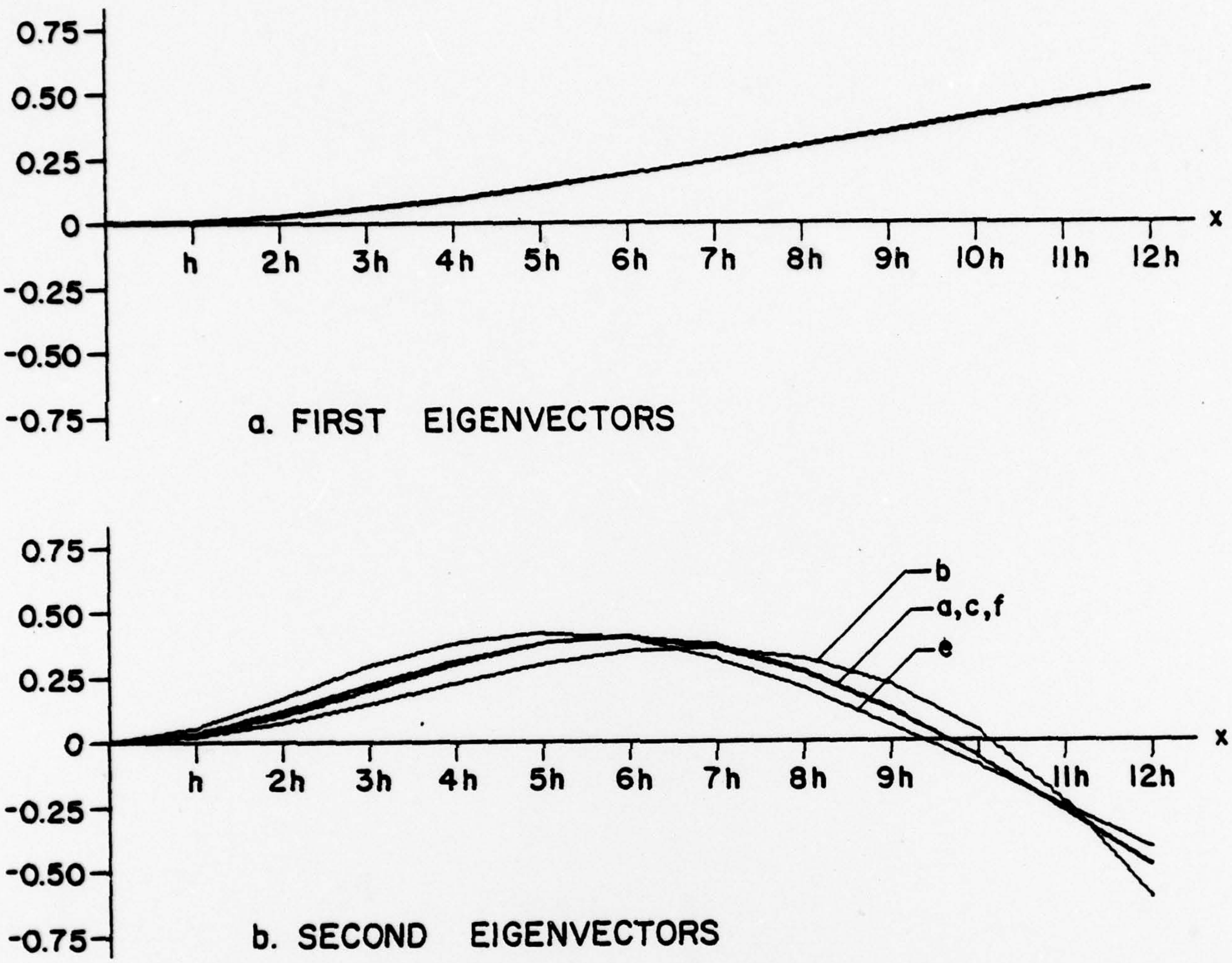


FIGURE 2 - EIGENVECTORS USING A TWO-TERM EXPANSION

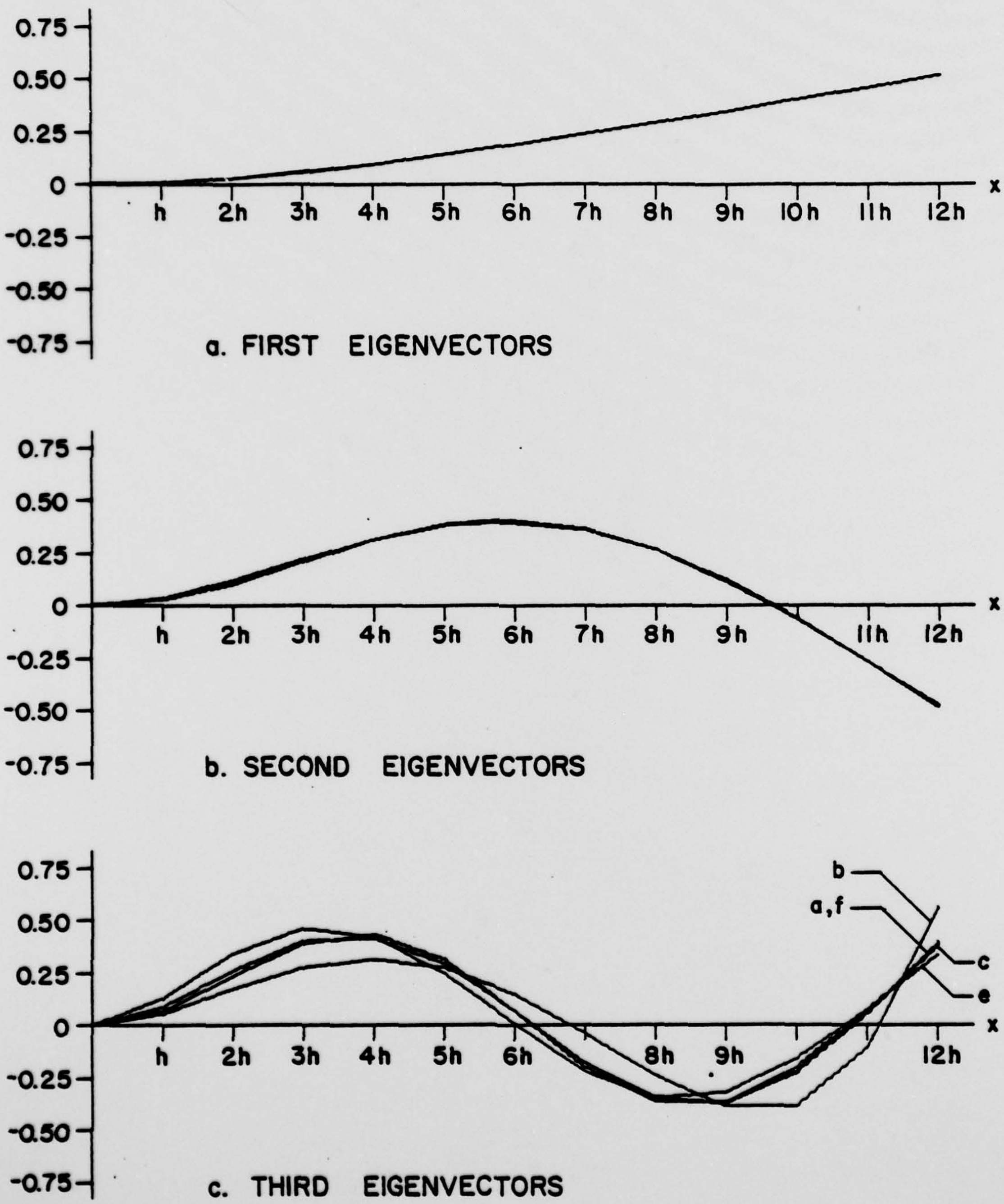


FIGURE 3 - EIGENVECTORS USING A THREE-TERM EXPANSION

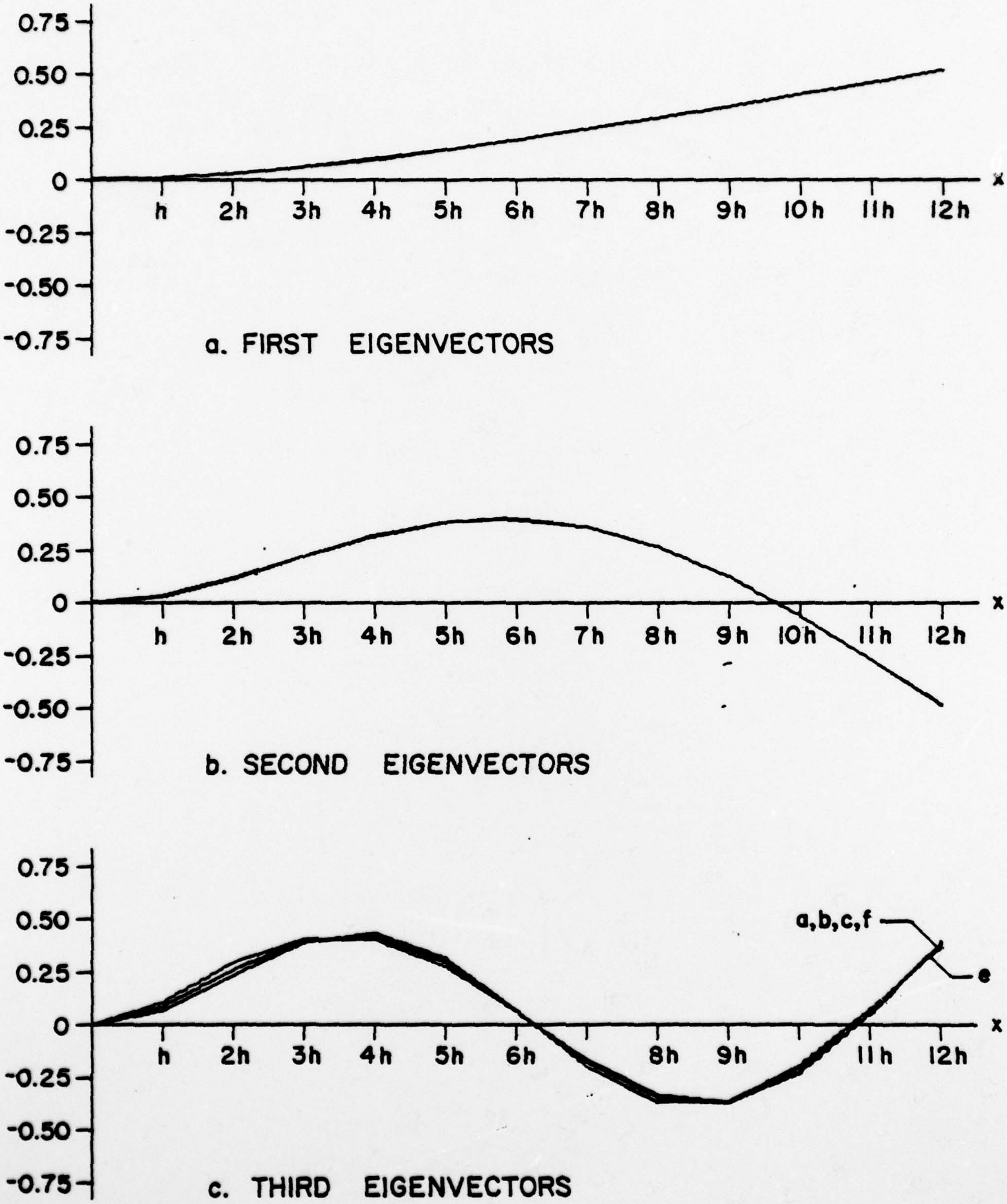
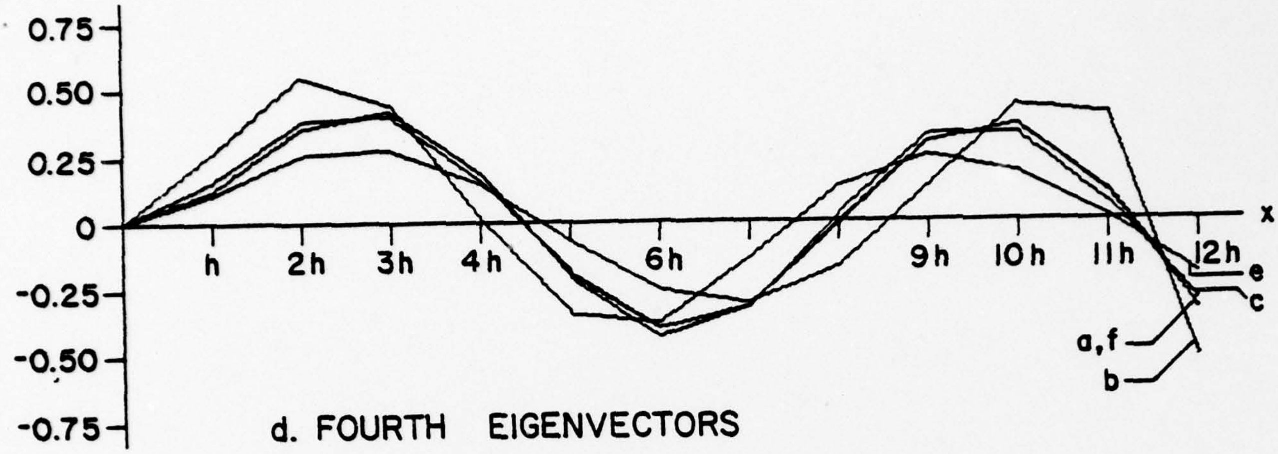


FIGURE 4 - EIGENVECTORS USING A FOUR-TERM EXPANSION



d. FOURTH EIGENVECTORS

FIGURE 4 - EIGENVECTORS USING A FOUR-TERM EXPANSION (continued)

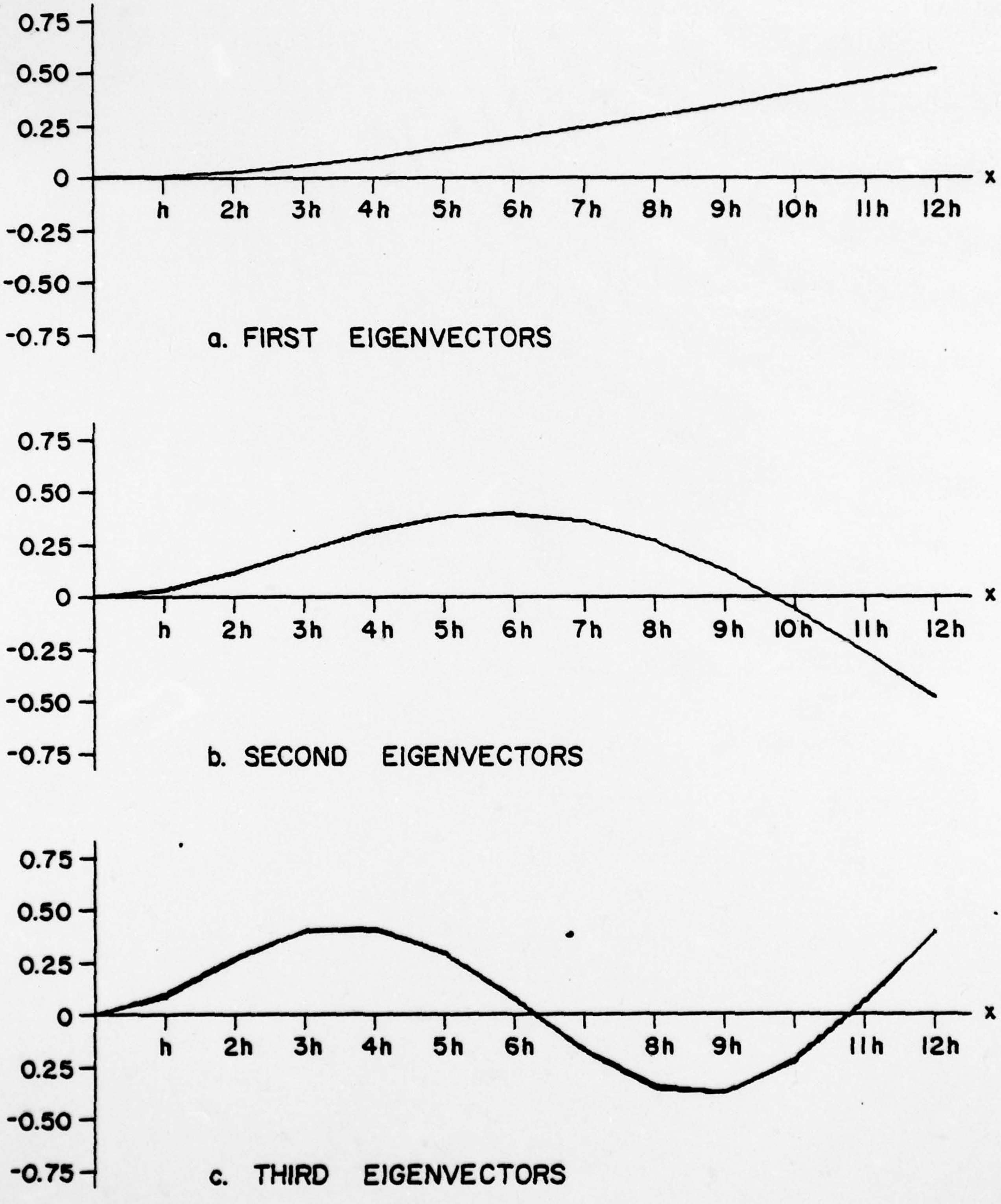


FIGURE 5 - EIGENVECTORS USING A FIVE-TERM EXPANSION

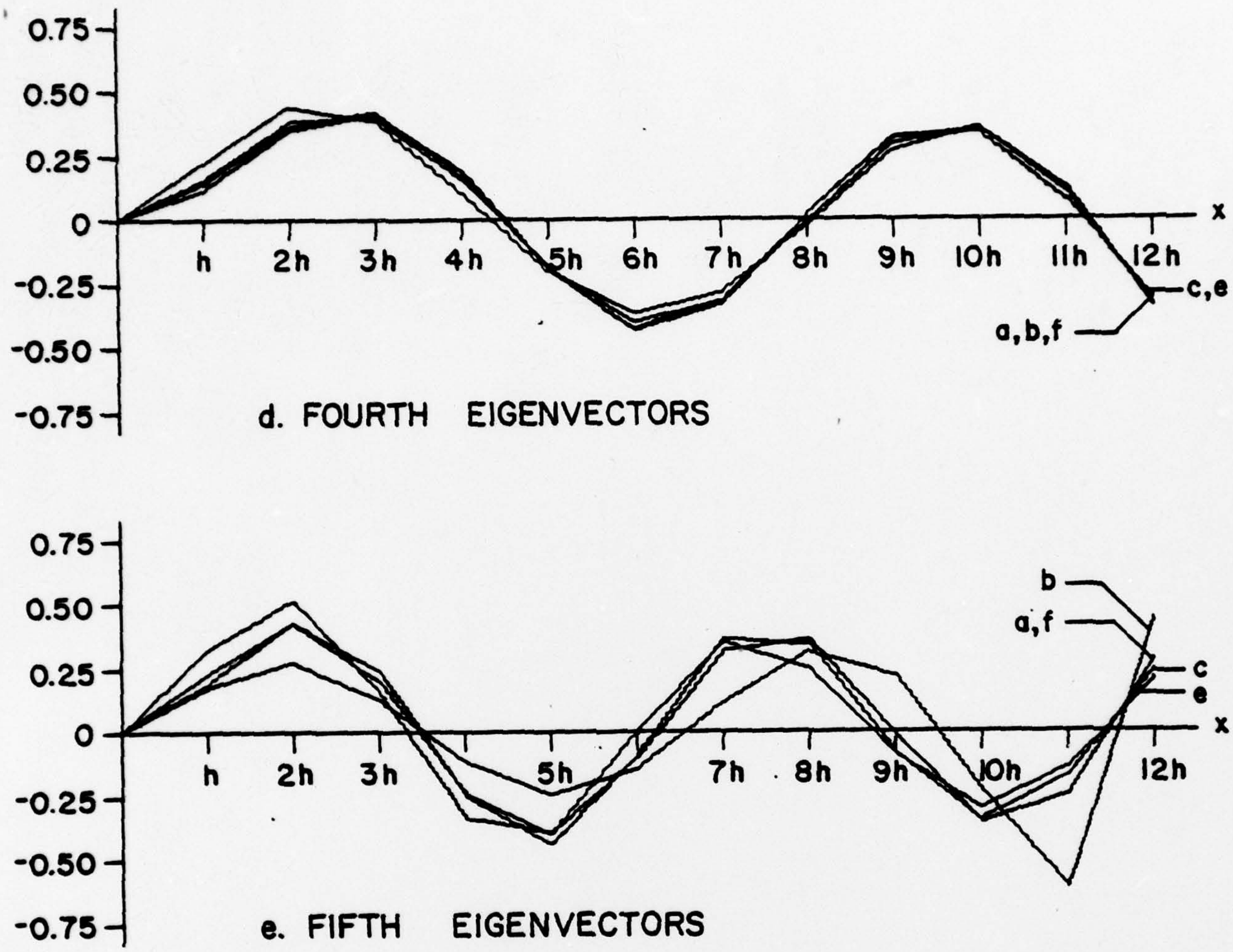


FIGURE 5 - EIGENVECTORS USING A FIVE-TERM EXPANSION (continued)